

Quasi-Perfect Lee Codes from Quadratic Curves over Finite Fields

Sihem Mesnager¹ Chunming Tang² Yanfeng Qi³

Abstract

Golomb and Welch conjectured in 1970 that there only exist perfect Lee codes for radius $t = 1$ or dimension $n = 1, 2$. It is admitted that the existence and the construction of quasi-perfect Lee codes have to be studied since they are the best alternative to the perfect codes. In this paper we firstly highlight the relationships between subset sums, Cayley graphs, and Lee linear codes and present some results. Next, we present a new constructive method for constructing quasi-perfect Lee codes. Our approach uses subsets derived from some quadratic curves over finite fields (in odd characteristic) to derive two classes of 2-quasi-perfect Lee codes are given over the space \mathbb{Z}_p^n for $n = \frac{p^k+1}{2}$ (with $p \equiv 1, -5 \pmod{12}$ and k is any integer, or $p \equiv -1, 5 \pmod{12}$ and k is an even integer) and $n = \frac{p^k-1}{2}$ (with $p \equiv -1, 5 \pmod{12}$, k is an odd integer and $p^k > 12$), where p is an odd prime. Our codes encompass the quasi-perfect Lee codes constructed recently by Camarero and Martínez. Furthermore, we solve a conjecture proposed by Camarero and Martínez (in "quasi-perfect Lee codes of radius 2 and arbitrarily large dimension", IEEE Trans. Inf. Theory, vol. 62, no. 3, 2016) by proving that the related Cayley graphs are Ramanujan or almost Ramanujan. The Lee codes presented in this paper have applications to constrained and partial-response channels, in flash memories and decision diagrams.

Index Terms

Lee distance, quasi-perfect codes, subset sums, Cayley graphs, Ramanujan graphs, quadratic curves.

I. INTRODUCTION

Throughout this paper, \mathbb{Z} and \mathbb{Z}_M denote the ring of integers and the ring of integer modulo M , respectively. By a *Lee code* \mathcal{C} of block size n over \mathbb{Z} (or \mathbb{Z}_M) we will understand a subset \mathcal{C} of the infinite lattice \mathbb{Z}^n (or the finite integer lattice \mathbb{Z}_M^n) endowed by the Lee distance. If \mathcal{C} has furthermore the structure of an additive group, then \mathcal{C} is called *Lee linear code*. Lee codes have many practical applications, in particular for the toroidal interconnection networks. They are also used for phase modulated and multilevel quantized-pulse-modulated channels (see e.g. [1], [2], [8], [12] and [19]). Moreover, it has been shown in [3], [4], [6], [9] and [10] that these codes are the foundation of designing placement strategies to distribute commonly shared resources like input/output (I/O) devices over a toroidal networks. Similar concepts can be used to design fault-tolerant techniques for this kind of networks [9]. Using space embeddings, Jiang et al. gave a method in [15] to construct Charge-Constrained Rank-Modulation codes (CCRM codes) from Lee error-correcting codes, which could be employed for flash memories. Astola and Stankovic considered in [7] Lee codes to build decision diagrams.

Perfect Lee codes are the most interesting and important subclass of Lee codes. Unfortunately, the perfect t -error correcting Lee codes of block length n over \mathbb{Z} , and over \mathbb{Z}_M , $M \geq 2n + 1$, shortly $PL(n, t)$ codes and $PL(n, t, M)$ codes, respectively, have been constructed only for $n = 1, 2$, and any t , and for $n \geq 3$ and $t = 1$. Moreover, as suggested by the well-known and long-standing conjecture of Golomb and Welch [13], $PL(n, t)$ codes and $PL(n, t, M)$, $M \geq 2n + 1$, codes do not exist in other cases. Despite the considerable amount of attempts in this topic, the conjecture is far being solved.

Although the Golomb-Welch conjecture has not been solved yet, its validity has been widely believed by the community. Therefore, failing finding perfect Lee codes, some codes which are "closed" to being perfect have been considered in the literature (see e.g. [5], where quasi-perfect codes over \mathbb{Z} and $QPL(n, t)$ codes have been introduced). Also, in [14] the authors presented some quasi-perfect codes for $n = 3$ and few radii. Later, Queiroz et al. characterized in [18] quasi-perfect codes over Gaussian and Eisenstein-Jacobi integers. As a consequence, linear quasi-perfect Lee codes were obtained for $n = 2$. In [11], from the quotient additive group of Gaussian integers, Camarero and Martínez constructed two-quasi-perfect Lee codes over the space \mathbb{Z}_p^n for p prime with $p \equiv \pm 5 \pmod{12}$, and $n = 2[\frac{p}{4}]$, where the notation $[\frac{p}{4}]$ stands for the closed integer to the rational number $\frac{p}{4}$.

In this manuscript, we firstly survey the connections between subset sums, Cayley graphs, and Lee linear codes. Next, we provide two classes of 2-quasi-perfect Lee codes over the space \mathbb{Z}_p^n for $n = \frac{p^k+1}{2}$ (with $p \equiv 1, -5 \pmod{12}$ and k is any integer, or $p \equiv -1, 5 \pmod{12}$ and k is an even integer) and $n = \frac{p^k-1}{2}$ (if $p \equiv -1, 5 \pmod{12}$, k is an odd integer and $p^k > 12$), where

This work was supported by the National Natural Science Foundation of China (Grant No. 11401480, 11531002). C. Tang also acknowledges support from 14E013 and CXTD2014-4 of China West Normal University. Y. Qi also acknowledges support from KSY075614050 of Hangzhou Dianzi University.

S. Mesnager is with Department of Mathematics, Universities of Paris VIII and XIII and Telecom ParisTech, LAGA, UMR 7539, CNRS, Sorbonne Paris Cité. e-mail: smesnager@univ-paris8.fr

C. Tang is with School of Mathematics and Information, China West Normal University, Nanchong, Sichuan, 637002, China. e-mail: tangchunming-math@163.com

Y. Qi is with School of Science, Hangzhou Dianzi University, Hangzhou, Zhejiang, 310018, China. e-mail: qiyanfeng07@163.com

p is an odd prime. Our classes are obtained by considering subsets derived from some quadratic curves over finite fields of odd characteristic. This paper generalizes the results of [11] since the 2-quasi-perfect Lee codes constructed by Camarero and Martínez in [11] correspond in fact to our codes when $p \equiv \pm 5 \pmod{12}$ and $k = 1$. Furthermore, This paper also proves that the related Cayley graphs are Ramanujan or almost Ramanujan. It solves a conjecture proposed by Camarero and Martínez [11].

This paper is organized as follows. In Section II, we recall and present some results (useful in the next sections) devoted to exponential sums, Lee linear codes and Cayley graphs. Next, in Section III, we survey the relationships between subset sums, Cayley graphs, and Lee linear codes. In Section IV, we introduce two classes of subsets of \mathbb{F}_{q^2} and \mathbb{F}_q^2 derived from quadratic curves defined over finite fields \mathbb{F}_q and provide several results related to their associate subset sums. In Section V, we show that the related graphs are Ramanujan or almost Ramanujan. which solves a conjecture proposed by Camarero and Martínez [11]. In Section VI, we present two infinite classes of 2-quasi-perfect Lee codes over finite alphabet.

II. PRELIMINARIES

We begin this section by fixing some notation which will be used throughout this paper.

- Given a set S , $\#S$ denotes its cardinality;
- \mathbb{Z} denotes the ring of integers;
- \mathbb{Z}_M denotes the ring of integers modulo M ;
- \mathbb{F}_q denotes the finite field with q elements (where q is a prime power p^k);
- \mathbb{F}_q^\times denotes the multiplicative group of \mathbb{F}_q ;
- Given a prime p , $p^* := (-1)^{\frac{p-1}{2}}p$;
- η and η_0 denote the quadratic characters of \mathbb{F}_q^\times and \mathbb{F}_p^\times , respectively;
- SQ_q and NSQ_q denote the set of all squares and nonsquares in \mathbb{F}_q^\times , respectively.
- $\zeta_p = e^{\frac{2\pi\sqrt{-1}}{p}}$ is the primitive p -th root of unity in the complex field \mathbb{C} .

A. Quadratic Reciprocity Law and exponential sums

Some known results related to particular cases of the well-known *Quadratic Reciprocity Law* ([16]) will be necessary for our results. We summarize those results the following proposition.

Proposition 2.1: If p is an odd prime greater than 3, then

- (1) $-1 \in \text{SQ}_p$ if and only if $p \equiv 1 \pmod{4}$.
- (2) $3 \in \text{SQ}_p$ if and only if $p \equiv \pm 1 \pmod{12}$.
- (3) $-3 \in \text{SQ}_p$ if and only if $p \equiv 1$ or $-5 \pmod{12}$.

We shall also use the following result related to quadratic exponential sums [16].

Lemma 2.2: Let $q = p^k$, with p an odd prime. Then, for any $a \in \mathbb{F}_q$, $c \in \mathbb{F}_q^\times$, we have

$$\sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^k(cx^2 + ax)} = (-1)^{k-1} \eta(c) \sqrt{p^*}^k \zeta_p^{\text{Tr}_1^k(-\frac{a^2}{4c})}.$$

Kloosterman sums form a special class of exponential sums. For $(a, b) \in \mathbb{F}_q \times \mathbb{F}_q^\times$, the Kloosterman sum $K_q(a, b)$ is defined by

$$K_q(a, b) = \sum_{x \in \mathbb{F}_q^\times} \zeta_p^{\text{Tr}_1^k(ax + \frac{b}{x})}.$$

Kloosterman sums are related to several mathematical and engineering problems and have been extensively studied in the literature. Unfortunately, it is very hard to evaluate Kloosterman sums. The *Hasse-Weil bound* on Kloosterman sums ([16]) is an estimation given by the following statement.

Theorem 2.3: Let $(a, b) \in \mathbb{F}_q^\times \times \mathbb{F}_q^\times$. Then $|K_q(a, b)| \leq 2\sqrt{q}$.

B. Lee linear codes

Since Lee codes over finite alphabet are the target of our study, the natural space to be considered is the vector space \mathbb{Z}_p^n over primitive finite fields. Therefore, a code \mathcal{C} will be a subset of \mathbb{Z}_p^n . This code is said to be linear if it is a subgroup of \mathbb{Z}_p^n . For $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_p^n$ their Lee distance is defined as

$$d_L(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^n \min\{|s| : s \equiv \mathbf{a}_i - \mathbf{b}_i \pmod{p}, s \in \mathbb{Z}\}.$$

The weight of a word \mathbf{c} is defined as its distance to the origin O , which will be denoted as $wt_L(\mathbf{c}) = d_L(\mathbf{c}, O)$. For any positive integer r , the Lee sphere of radius r is defined as all the points whose weight is less or equal to r , that is:

$$B_r^n = \{\mathbf{c} : wt_L(\mathbf{c}) \leq r\}.$$

Note that, when $p \geq 5$, then, for any dimension $n \geq 1$, we have $\#B_2^n = 2n^2 + 2n + 1$ ([13]).

A code \mathcal{C} is said to be t -error correcting if t is the greatest integer such that for any word \mathbf{w} there is at the most one codeword $\mathbf{c} \in \mathcal{C}$ with $d_L(\mathbf{w}, \mathbf{c}) \leq t$. Thus, t is called the *error correction* of \mathcal{C} . A code \mathcal{C} is said to be r -covering if r is the smallest integer such that for any word \mathbf{w} there is at the least one codeword $\mathbf{c} \in \mathcal{C}$ with $d_L(\mathbf{w}, \mathbf{c}) \leq r$. Thus, r denotes the covering radius of \mathcal{C} . A code that is both t -error correcting and t -covering is said to be *perfect*. A code that is t -error correcting and $(t+1)$ -covering is said to be *t -quasi-perfect*. Golomb and Welch conjectured in [13] that there only exist perfect Lee codes for $t = 1$ or $n = 1, 2$. Therefore, the existence of quasi-perfect codes must be studied since they are the best alternative to the perfect codes.

Linear codes can be constructed from Abelian groups and their special subsets. In the following, Γ denotes an Abelian group with exponential p , that is, for any $\beta \in \Gamma$, $p \cdot \beta = 0$. Thus, Γ can be considered as a vector space over \mathbb{F}_p . Set $m := \dim_{\mathbb{F}_p} \Gamma$ and let H be a set of generators of Γ , with $H = -H$ and $0 \notin H$. Thus, the cardinality $\#H$ of H must be even. By setting, $\#H = 2n$ one has $H = \{\pm\beta_1, \pm\beta_2, \dots, \pm\beta_n\}$ and the associated linear code is

$$\mathcal{C}(\Gamma; H) = \{(c_1, \dots, c_n) \in \mathbb{F}_p^n : c_1\beta_1 + \dots + c_n\beta_n = 0\}.$$

The dimension of $\mathcal{C}(\Gamma; H)$ over \mathbb{F}_p equals $n - m$.

C. Cayley graph

A graph X is a triple consisting of a vertex set $V = V(X)$, an edge set $E = E(X)$ and a map that associates with each edge two vertices (not necessarily distinct) called its *endpoints*. The degree of a vertex v is the number of edges incident with v . A graph is called k -regular if every vertex has degree k . Now, let us recall some basic definitions. The distance $d_X(x, y)$ between two vertices x, y in a graph X is defined as the number of edges in the shortest path from x to y . The diameter of a graph X is the maximum among distances between every pair of vertices.

To any graph, one can associate the *adjacency matrix* A which is an $N \times N$ matrix (where $N = \#V$) with rows and columns indexed by the elements of the vertex set and the (x, y) -th entry is the number of edges connecting x and y . Since our graphs are undirected, the matrix A is symmetric. Consequently, all of its eigenvalues are real. *Ramanujan graphs* are good expander graphs that attain the spectral bound ([17]). More specifically, a k -regular graph X is a Ramanujan graph if and only if, for every eigenvalue λ of its adjacency matrix it holds either $|\lambda| = k$ or $|\lambda| \leq 2\sqrt{k-1}$.

There is a simple procedure for constructing regular graphs using group theory. This can be described as follows. Let Γ and H be defined as the previous subsection. Now construct the Cayley graph $\text{Cay}(\Gamma; H)$ by having the vertex set to be the elements of Γ with (α, β) an edge if and only if $\beta - \alpha \in H$. Then, $\text{Cay}(\Gamma; H)$ is a $\#H$ -regular graph. Then the *error correction capacity* of the Cayley graph $\text{Cay}(\Gamma; H)$ is defined as the greatest integer t such that for every vertex x there are $\#B_t^n$ vertices at distance t or less from x , where $\#H = 2n$. Note that Cayley graph is vertex-transitive and therefore it is enough to count the number of vertices around one vertex to determine its error correction capacity and the diameter can also be calculated as the maximum distance to one particular vertex, usually $0 \in \Gamma$.

The eigenvalues of the Cayley graph can be determined by the following theorem given in [17].

Theorem 2.4: Let Γ be a finite Abelian group and H a subset of Γ with $H = -H$ and $0 \notin H$. The eigenvalues of the adjacency matrix of $\text{Cay}(\Gamma; H)$ are given by

$$\lambda_\chi = \sum_{\beta \in H} \chi(\beta),$$

where χ ranges over all characters of Γ .

Remark Let $\#H = 2n$. Notice that for the trivial character χ_0 , we have $\lambda_{\chi_0} = 2n$. For all $\chi \neq 1$, one has

$$|\sum_{\beta \in H} \chi(\beta)| < k.$$

Then, the graph is connected. Thus, to construct Ramanujan graphs, we require

$$|\sum_{\beta \in H} \chi(\beta)| \leq 2\sqrt{\#H - 1}$$

for every non-trivial character χ of Γ .

III. SUBSET SUMS, CAYLEY GRAPH AND LEE CODE

The correspondence between subset sums, linear codes and Cayley graph is explained in this section. Firstly, we start by stating some fundamental definitions on subset sums. For any subsets A and B of an Abelian group, define $-A = \{-x : x \in A\}$, $A + B = \{x + y : x \in A, y \in B\}$ and $A^{(i+1)} = A^{(i)} + A$ with $A^{(1)} = A$ and $A^{(0)} = \{0\}$.

We introduce the following notions.

Definition Given a finite Abelian group Γ and a set of generators $H = \{\pm\beta_1, \dots, \pm\beta_n\}$ with cardinality $\#H = 2n$. The *expansion critical index* of H is defined as the greatest integer t such that $\#(\cup_{i=0}^t H^{(i)}) = \#B_t^n$. The *expansion limit index* of H is defined as the smallest integer r such that $\cup_{i=0}^r H^{(i)} = \Gamma$.

Theorem 3.1: Let $p \geq 5$ be an odd prime, Γ be a finite Abelian group with exponential p and $H = \{\pm\beta_1, \dots, \pm\beta_n\}$ be a set of generators with $\#H = 2n$. Then, the following three quantities coincide,

- (1) the expansion critical index of subset H ;
- (2) the error correction capacity of Cayley graph $\text{Cay}(\Gamma; H)$;
- (3) the error correction of Lee linear code $\mathcal{C}(\Gamma; H)$.

Proof: Let D_i be the set of vertices at distance i from 0. Then, from the definition of Cayley graph $\text{Cay}(\Gamma; H)$, the vertices at distance t or less from 0 is the set $\cup_{i=0}^t D_i = \cup_{i=0}^t H^{(i)}$. Thus, the expansion critical index of subset H equals the error correction capacity of Cayley graph $\text{Cay}(\Gamma; H)$. By Theorem 4 in [11], the error correction capacity of Cayley graph $\text{Cay}(\Gamma; H)$ coincides with the error correction of Lee linear code $\mathcal{C}(\Gamma; H)$, which completes the proof. ■

Theorem 3.2: Let $p \geq 5$ be an odd prime, Γ be a finite Abelian group with exponential p and $H = \{\pm\beta_1, \dots, \pm\beta_n\}$ be a set of generators with $\#H = 2n$. Then, the following three quantities coincide,

- (1) the expansion limit index of subset H ;
- (2) the diameter of Cayley graph $\text{Cay}(\Gamma; H)$;
- (3) the covering radius of Lee linear code $\mathcal{C}(\Gamma; H)$.

Proof: Let D_i be defined as in the proof of Theorem 3.1 and the expansion limit index of subset H is r . Then, the diameter of Cayley graph $\text{Cay}(\Gamma; H)$ is less than or equal to r . From the definition of expansion limit index, $(\cup_{i=0}^r H^{(i)}) \setminus (\cup_{i=0}^{r-1} H^{(i)}) \neq \emptyset$. The vertices in $(\cup_{i=0}^r H^{(i)}) \setminus (\cup_{i=0}^{r-1} H^{(i)})$ has distance r from 0. Hence, the diameter of Cayley graph $\text{Cay}(\Gamma; H)$ is equal to r . By Theorem 4 in [11], the diameter of Cayley graph $\text{Cay}(\Gamma; H)$ coincides with the covering radius of Lee linear code $\mathcal{C}(\Gamma; H)$, which completes the proof. ■

Theorem 3.3: Let $p \geq 5$ be an odd prime, Γ be a finite Abelian group with exponential p and $H = \{\pm\beta_1, \dots, \pm\beta_n\}$ be a set of generators with $\#H = 2n$. If $2n^2 + 2n + 1 < \#\Gamma < \frac{1}{3}(1 + 2n)(3 + 2n + 2n^2)$, $\#(\cup_{i=0}^2 H^{(i)}) = 2n^2 + 2n + 1$ and $\cup_{i=0}^3 H^{(i)} = \Gamma$. Then, the expansion critical index and the expansion limit index of subset H are 2 and 3, respectively.

Proof: Note that the n -dimensional sphere of radius 2 has cardinality $\#B_2^n = 2n^2 + 2n + 1$ and the n -dimensional sphere of radius 3 has cardinality $\#B_3^n = \frac{1}{3}(1 + 2n)(3 + 2n + 2n^2)$. From the definitions of expansion critical index and expansion limit index, the expansion critical index and the expansion limit index of subset H are 2 and 3, respectively. This completes the proof. ■

From Theorems 3.3, 3.1 and 3.2, we derive the following statement.

Theorem 3.4: Let $p \geq 5$ be an odd prime, Γ be a finite Abelian group with exponential p and $H = \{\pm\beta_1, \dots, \pm\beta_n\}$ be a set of generators with $\#H = 2n$. If $2n^2 + 2n + 1 < \#\Gamma < \frac{1}{3}(1 + 2n)(3 + 2n + 2n^2)$, $\#(\cup_{i=0}^2 H^{(i)}) = 2n^2 + 2n + 1$ and $\cup_{i=0}^3 H^{(i)} = \Gamma$. Then, $\mathcal{C}(\Gamma; H)$ is a linear 2-quasi-perfect p -ary Lee code over \mathbb{F}_p^n with dimension $n - \dim_{\mathbb{F}_p} \Gamma$.

The next theorem follows from Theorems 3.3, 3.1 and 3.2.

Theorem 3.5: Let $p \geq 5$ be an odd prime, Γ be a finite Abelian group with exponential p and $H = \{\pm\beta_1, \dots, \pm\beta_n\}$ be a set of generators with $\#H = 2n$. If $\#\Gamma = 2n^2 + 2n + 1$ and $\cup_{i=0}^2 H^{(i)} = \Gamma$. Then, $\mathcal{C}(\Gamma; H)$ is a linear 2-perfect p -ary Lee code over \mathbb{F}_p^n with dimension $n - \dim_{\mathbb{F}_p} \Gamma$.

IV. SUBSETS AND SUBSET SUMS FROM QUADRATIC CURVES

Let $q = p^k$ with p an odd prime, and δ be a quadratic nonresidue in \mathbb{F}_q . Then $\mathbb{F}_q[\sqrt{\delta}]$ is an extension of \mathbb{F}_q with degree 2, which is denoted by \mathbb{F}_{q^2} . For any $z = x + \sqrt{\delta}y$ ($x, y \in \mathbb{F}_q$), define $\bar{z} = x - \sqrt{\delta}y$ and $\mathcal{N}(z) = z \cdot \bar{z} = x^2 - \delta y^2$. Then $\mathcal{N}(\cdot)$ is a surjective morphism from $\mathbb{F}_{q^2}^\times$ to \mathbb{F}_q^\times . Define a subset of \mathbb{F}_{q^2}

$$H_+ = \{z \in \mathbb{F}_{q^2} : \mathcal{N}(z) = 1\}. \quad (1)$$

Then $\#H_+ = q + 1$ and H_+ can also be view as the set of points on the following quadratic curve over \mathbb{F}_q :

$$x^2 - \delta y^2 = 1.$$

We will also considering the following subset of \mathbb{F}_q^\times :

$$H_- = \{(x, \frac{1}{x}) : x \in \mathbb{F}_q^\times\}. \quad (2)$$

Then H_- is the set of points on the quadratic curve

$$xy = 1.$$

In the following subsections, we will give some properties of two subsets H_+ and H_- .

A. Some results on subset sums for H_+

Lemma 4.1: Let $\delta \in \text{NSQ}_q$ and $c \in \mathbb{F}_q^\times$. Then $\#\{(x, y) : x^2 - \delta y^2 = c\} = q + 1$.

Proof: Note that $x^2 - \delta y^2 = c$ if and only if $\mathcal{N}(x + \sqrt{\delta}y) = c$. Hence, this lemma follows. \blacksquare

Lemma 4.2: Let $c \in \mathbb{F}_q^\times$. Define the set $D_c = \{x \in \mathbb{F}_q : x^2 = c \text{ or } x^2 - c \in \text{NSQ}_q\}$. Then $\#D_c = \begin{cases} \frac{q+3}{2}, & c \in \text{SQ}_q \\ \frac{q+1}{2}, & c \in \text{NSQ}_q \end{cases}$.

Proof: Suppose $c \in \text{SQ}_q$. From the definition D_c , we have

$$D_c = \{x \in \mathbb{F}_q : x^2 - \delta y^2 = c\}.$$

Let $\#D_c = s$ and $D_c = \{\sqrt{c}, -\sqrt{c}, x_1, \dots, x_{s-2}\}$. Then all the points on $x^2 - \delta y^2 = c$ are

$$(\sqrt{c}, 0), (-\sqrt{c}, 0), (x_1, \pm y_1), \dots, (x_{s-2}, \pm y_{s-2}),$$

where $y_i \in \mathbb{F}_q^\times$ and $x_i^2 - \delta y_i^2 = c$. Hence, we have

$$\#\{(x, y) : x^2 - \delta y^2 = c\} = 2(s - 2) + 2 = 2s - 2.$$

From Lemma 4.1, $2s - 2 = q + 1$ and $s = \frac{q+3}{2}$.

Suppose $c \in \text{NSQ}_q$. Let $D_c = s$ and $D_c = \{x_1, \dots, x_s\}$. Then all the points on $x^2 - \delta y^2 = c$ are

$$(x_1, \pm y_1), \dots, (x_s, \pm y_s),$$

where $y_i \in \mathbb{F}_q^\times$ and $x_i^2 - \delta y_i^2 = c$. Hence, $\#\{(x, y) : x^2 - \delta y^2 = c\} = 2s$. From Lemma 4.1, we have $2s = q + 1$ and $s = \frac{q+1}{2}$. \blacksquare

Lemma 4.3: Let $w \in \mathbb{F}_{q^2}^\times$, $c = \mathcal{N}(w)$, and $I_w = \{\mathcal{N}(z) : z \in H_+ + H_+w\}$. Then

- (1) $\#I_w = \begin{cases} \frac{q+3}{2}, & c \in \text{SQ}_q \\ \frac{q+1}{2}, & c \in \text{NSQ}_q \end{cases}$.
- (2) $1 \in I_1$ if and only if $-3 \in \text{NSQ}_q$ or $p = 3$.

Proof: (1) Let $z_1, z_2 \in H_+$. Then $\mathcal{N}(z_1 + z_2w) = \mathcal{N}(z_1(1 + \frac{z_2}{z_1}w)) = \mathcal{N}(z_1)\mathcal{N}(1 + \frac{z_2}{z_1}w)$. From the definition of H_+ , we have

$$I_w = \{\mathcal{N}(1 + z) : z \in H_+w\}.$$

From the definition of $\mathcal{N}(\cdot)$, $t \in I_w$ if and only if the following system of equations has solutions:

$$\begin{cases} x^2 - \delta y^2 = c \\ (1 + x)^2 - \delta y^2 = t. \end{cases}$$

This system of equations is equivalent to

$$\begin{cases} x^2 - \delta y^2 = c \\ x = \frac{t-c-1}{2}. \end{cases}$$

Hence, $t \in I_w$ if and only if the equation

$$\delta y^2 = \frac{(t - c - 1)^2}{4} - c, \tag{3}$$

with the variable y has solutions, that is, $\frac{t-c-1}{2} \in D_c$. From Lemma 4.2, we have

$$\#I_w = \begin{cases} \frac{q+3}{2}, & c \in \text{SQ}_q \\ \frac{q+1}{2}, & c \in \text{NSQ}_q \end{cases}.$$

(2) From the Equation (3), we have $1 \in I_1$ if and only if, $-\frac{3}{4} \in \text{NSQ}_q$ or $-3 = 0$. Hence, $1 \in I_1$ if and only if, $-3 \in \text{NSQ}_q$ or $p = 3$. \blacksquare

Lemma 4.4: Let $w \in \mathbb{F}_{q^2}^\times$, where $w \notin H_+$. Then

$$\#(H_+ + H_+w) = \begin{cases} \frac{(q+1)(q+3)}{2}, & \mathcal{N}(w) \in \text{SQ}_q \\ \frac{(q+1)^2}{2}, & \mathcal{N}(w) \in \text{NSQ}_q \end{cases}.$$

Proof: Since $w \notin H_+$, $0 \notin H_+ + H_+w$. From Lemma 4.3, this lemma follows. \blacksquare

Theorem 4.5: Let p be an odd prime. Then

- (1) $\#H_+^{(2)} = 1 + \frac{1}{2}(q+1)^2$.
- (2) $\#H_+^{(2)} \setminus (H_+ \cup \{0\}) = \begin{cases} \frac{(q+1)^2}{2}, & -3 \in \text{SQ}_q \\ \frac{(q-1)(q+1)}{2}, & -3 \in \text{NSQ}_q \text{ or } p = 3 \end{cases}$.

Proof: (1) For any $z_1 \in H_+$ and $z_2 \in H_+^{(2)}$, we have $z_1 z_2 \in H_+^{(2)}$. Thus, one has the following decomposition

$$H_+^{(2)} = \cup_{t \in I_1 \setminus \{0\}} H_+ t \cup \{0\}.$$

By Lemma 4.3,

$$\#H_+^{(2)} = 1 + (q+1)\frac{q+1}{2} = 1 + \frac{1}{2}(q+1)^2.$$

(2) If $-3 \in \text{NSQ}_q$ or $p=3$, from Lemma 4.3, $1 \in I_1$. Then $H_+ \cdot 1 \subseteq H_+^{(2)}$. Hence, $\#(H_+^{(2)} \setminus (H_+ \cup \{0\})) = 1 + \frac{1}{2}(q+1)^2 - (q+1) - 1$, that is, $\#(H_+^{(2)} \setminus (H_+ \cup \{0\})) = \frac{(q+1)(q-1)}{2}$.

If $-3 \in \text{SQ}_q$, from Lemma 4.3, $1 \notin I_1$. From the statement (1) of this lemma, we have $\#(H_+^{(2)} \setminus (H_+ \cup \{0\})) = \frac{1}{2}(q+1)^2$. ■

Theorem 4.6: Let p be an odd prime. Then $H_+^{(3)} \cup \{0\} = \mathbb{F}_{q^2}$.

Proof: Suppose that there exists $z \in \mathbb{F}_{q^2}^\times$, with $z \notin H_+^{(3)}$. Then $H_+ z \cap H_+^{(3)} = \emptyset$. With the assumption, we first prove that $(H_+ + H_+ z) \cap (H_+ + H_+) = \emptyset$. If there exist $u_i \in H_+ (i = 1, 2, 3, 4)$ such that

$$u_1 + u_2 z = u_3 + u_4,$$

then $z = \frac{u_3}{u_2} + \frac{u_4}{u_2} + (-\frac{u_1}{u_2}) \in H_+^{(3)}$. This contradicts $z \notin H_+^{(3)}$. Hence, $(H_+ + H_+ z) \cap (H_+ + H_+) = \emptyset$. Then $q^2 = \#\mathbb{F}_{q^2} \geq \#(H_+ + H_+ z) + \#(H_+ + H_+)$. From Lemma 4.4 and Theorem 4.5, we have

$$q^2 \geq \frac{(q+1)^2}{2} + 1 + \frac{1}{2}(q+1)^2 = 1 + (q+1)^2.$$

This makes a contradiction. Hence, this theorem follows. ■

Theorem 4.7: Let p be an odd prime with $p \equiv 1$ or $-5 \pmod{12}$ and k be any positive integer, or $p \geq 5$ and k be an even positive integer. Let H_+ be defined as (1). Then, the expansion critical index and the expansion limit index of subset H_+ are equal 2 and 3, respectively.

Proof: Note that $H_+ = q+1$. If p is an odd prime with $p \equiv 1$ or $-5 \pmod{12}$ and k any positive integer, or $p \geq 5$ and k an even positive integer, then $-3 \in \text{SQ}_q$. From Theorem 4.5 and Theorem 3.3, the expansion critical index equals 2. By Theorem 4.6 and Theorem 3.3, the expansion limit index equals 3. This completes the proof. ■

B. Some results on subset sums for H_-

Lemma 4.8: let p be an odd prime, $(a, b) \in \mathbb{F}_q^2$, and $t = ab$, where $(a, b) \neq (0, 0)$. Then $(a, b) \in H_- + H_-$ if and only if $t \neq 0$ and $(\frac{t}{2} - 1)^2 - 1 \in \text{SQ}_q \cup \{0\}$.

Proof: Obviously, for $(a, b) \in H_-^{(2)} \setminus \{(0, 0)\}$, we have $t = ab \neq 0$. From the definition of $H_-^{(2)}$, we have $(a, b) \in H_-^{(2)} \setminus \{(0, 0)\}$ if and only if the following system of equations

$$\begin{cases} x + y = a \\ \frac{1}{x} + \frac{1}{y} = b \end{cases} \quad (4)$$

has solutions. This is equivalent to the following equation

$$\lambda^2 - a\lambda + \frac{a}{b} = 0.$$

has solution. Since the determinant of this quadratic equation is $\Delta = a^2 - \frac{4a}{b} = \frac{ab}{b^2}(ab - 4) = \frac{4((\frac{t}{2}-1)^2-1)}{b^2}$, this quadratic equation has solutions if and only if $(\frac{t}{2} - 1)^2 - 1 \in \text{SQ}_q \cup \{0\}$. From the above discussion, this lemma follows. ■

Lemma 4.9: Let p be an odd prime greater than 3. Then $\#\{ab : (a, b) \in H_-^{(2)} \setminus \{(0, 0)\}\} = \frac{q-1}{2}$.

Proof: From Lemma 4.2, we have

$$\#\{x \in \mathbb{F}_q : x^2 - 1 \in \text{SQ}_q \cup \{0\}\} = \frac{q+1}{2}.$$

From Lemma 4.8, we have

$$\#\{ab : (a, b) \in H_-^{(2)} \setminus \{(0, 0)\}\} = \frac{q-1}{2}.$$

Theorem 4.10: Let p be an odd prime. Then

(1) $\#H_-^{(2)} = 1 + \frac{(q-1)^2}{2}$;

(2) $\{0\} \in H_-^{(2)}$ and $H_- \cap H_-^{(2)} = \begin{cases} \emptyset, & -3 \in \text{NSQ}_q \\ H_-, & -3 \in \text{SQ}_q \text{ or } p = 3 \end{cases}$;

(3) $\#(H_-^{(2)} \setminus (H_- \cup \{0\})) = \begin{cases} \frac{(q-1)^2}{2}, & -3 \in \text{NSQ}_q \\ \frac{(q-1)^2}{2} - (q-1), & -3 \in \text{SQ}_q \text{ or } p = 3 \end{cases}$.

Proof: (1) From Lemma 4.8 and Lemma 4.9, this statement follows.

(2) From $(0, 0) = (1, 1) + (-1, -1)$, we have $(0, 0) \in H_-^{(2)}$. Obviously, $H_- \cap H_-^{(2)} = \emptyset$ or H_- . Further, $H_- \cap H_-^{(2)} = H_-$ if and only if $(1, 1) \in H_-^{(2)}$. From Lemma 4.8, $(1, 1) \in H_-^{(2)}$ if and only if $-3 \in \text{SQ}_q \cup \{0\}$. Hence this statement follows. ■

(3) From the statements (1) and (2), this statement follows. \blacksquare

Lemma 4.11: If $t \neq -1$, the polynomial $P_t(x, y) = (xy - x - y)(x + y + t) + xy \in \mathbb{F}_q[x, y]$ is absolutely irreducible.

Proof: Suppose that the polynomial $P_t(x, y)$ is not absolute irreducible. Then there exist polynomials $A(x, y)$ and $B(x, y)$ with coefficients in the algebraic closure of \mathbb{F}_q such that $P_t(x, y) = AB$, where $\deg(A) = 1$ and $\deg(B) = 2$. Further, the product of the leading terms of A and B is $xy(x + y)$. Then we consider the following cases:

1) Case $A(x, y) = x + b$. From the symmetry $P_t(x, y) = P_t(y, x)$, we have $y + b$ is also a factor of $P_t(x, y)$. Hence, $\frac{P_t(x, y)}{(x+b)(y+b)}$ is a factor of $P_t(x, y)$ of the form $x + y + a$;

2) Case $A(x, y) = y + b$. From the similar discuss, $P_t(x, y)$ has a factor of the form $x + y + a$;

3) Case $A(x, y) = x + y + a$.

From the above discussion, $x + y + a$ must be a factor of $P_t(x, y)$. Then

$$P_t(x, y) = (xy - x - y)(x + y + t) + xy \equiv 0 \pmod{x + y + a}.$$

We have

$$(t - a + 1)xy + a(t - a) \equiv 0 \pmod{x + y + a}.$$

Hence, $t - a + 1 = 0$ and $a(t - a) = 0$. Hence, $a = 0$ and $t = -1$. It makes a contradiction. Hence, this lemma follows. \blacksquare

Theorem 4.12: Let q be a power of odd prime with $q \geq 13$. Then $H_-^{(3)} \cup \{(0, 0)\} = \mathbb{F}_q^2$.

Proof: Obviously, $H_- \subseteq H_-^{(3)}$. For any $(a, b) \in \mathbb{F}_q^2 \setminus \{(0, 0)\}$, one has $a \neq 0$ or $b \neq 0$. Suppose that $b \neq 0$. Then $(a, b) \in H_-^{(3)}$ if and only if the following system of equations has solutions:

$$\begin{cases} x + y + z = a \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = b. \end{cases} \quad (5)$$

We set $\frac{1}{0} = 0$. This system of equations is equivalent to the following system of equations:

$$\begin{cases} bx + by + bz = ab \\ \frac{1}{bx} + \frac{1}{by} + \frac{1}{bz} = 1. \end{cases}$$

Hence, $(a, b) \in H_-^{(3)}$ if and only if $(ab, 1) \in H_-^{(3)}$. Hence, we just need to prove that $(-t, 1) \in H_-^{(3)}$ for any $t \in \mathbb{F}_q$. If $t = -1$, $(1, 1) = (1, 1) + (1, 1) + (-1, -1) \in H_-^{(3)}$. If $t \neq -1$, then $(-t, 1) \in H_-^{(3)}$ if and only if the following system of equations has solutions:

$$\begin{cases} x + y + z = -t \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1. \end{cases}$$

From this system of equations, we have

$$\frac{1}{x} + \frac{1}{y} - \frac{1}{x + y + t} = 1.$$

Further, we obtain

$$(x + y + t)(xy - x - y) + xy = 0.$$

This equation has solutions if and only if the following affine curve

$$E_t : (x + y + t)(xy - x - y) + xy = 0,$$

defined over \mathbb{F}_q has rational points. The correspondence projective curve are

$$\overline{E}_t : (X + Y + tZ)(XY - ZX - YZ) + XYZ = 0.$$

Then, we denote the \mathbb{F}_q -rational points sets associated with E_t and \overline{E}_t respectively by

$$E_t(\mathbb{F}_q) = \{(x, y) \in \mathbb{F}_q^2 : (x + y + t)(xy - x - y) + xy = 0\}$$

and

$$\overline{E}_t(\mathbb{F}_q) = \{(X : Y : Z) \in \mathbb{P}_{\mathbb{F}_q}^2 : (X + Y + tZ)(XY - ZX - YZ) + XYZ = 0\},$$

where $\mathbb{P}_{\mathbb{F}_q}^2$ denotes the projective space of dimension 2 over \mathbb{F}_q . The notation $(X : Y : Z)$ indicates a projective point, which is the same point as $(\lambda X : \lambda Y : \lambda Z)$ for any $\lambda \neq 0$. Thus, affine solutions can be recovered by taking $\lambda = Z^{-1}$; except for solutions $(X : Y : 0)$, which are the points at the infinite.

From Lemma 4.11, \overline{E}_t is an absolutely irreducible polynomial curves of degree 3. Hasse-Weil bound states that

$$|\#\overline{E}_t(\mathbb{F}_q) - (q + 1)| \leq 2\sqrt{q}. \quad (6)$$

The only three projective points $(X : Y : Z)$ with $Z = 0$ are $\{(1 : 0 : 0), (0 : 1 : 0), (1 : -1 : 0)\}$. The only three affine points with $xy(x + y + t) = 0$ are points in the set $G_t = \{(0, 0), (0, -t), (-t, 0)\}$. Hence,

$$\#(E_t(\mathbb{F}_q) \setminus G_t) = \#(\overline{E}_t(\mathbb{F}_q)) - 6.$$

From Equation (6), if $q \geq 13$, we have $\#(\overline{E}_t(\mathbb{F}_q)) - 6 > 0$. Hence, $(-t, 1) \in H_-^{(3)}$, which concludes the proof. \blacksquare

Remark For $q = 3, 5, 7, 9$, or 11 , we can verify that $H_-^{(3)} \cup \{(0, 0)\} \subsetneq \mathbb{F}_q^2$.

Theorem 4.13: Let p be an odd prime with $p \equiv -1$ or $5 \pmod{12}$, k be odd integer, and $q = p^k > 12$. Let H_- be defined as (2). Then, the expansion critical index and the expansion limit index of subset H_- are equal to 2 and 3, respectively.

Proof: Note that $H_- = q - 1$. If p is an odd prime with $p \equiv -1$ or $5 \pmod{12}$ and k odd integer, then $-3 \in \text{NSQ}_q$. From Theorem 4.10 and Theorem 3.3, the expansion critical index is 2. By Theorem 4.12 and Theorem 3.3, the expansion limit index is 3. This completes the proof. \blacksquare

V. CAYLEY GRAPHS FROM H_+ AND H_-

In this section, we will determined the eigenvalues of the Cayley graphs $\text{Cay}(\mathbb{F}_{q^2}; H_+)$ and $\text{Cay}(\mathbb{F}_q^2; H_-)$. Then, we will show the related graphs are Ramanujan or almost Ramanujan. It solves a conjecture proposed by Camarero and Martínez [11].

A. Cayley graph $\text{Cay}(\mathbb{F}_{q^2}; H_+)$

Theorem 5.1: Let p be an odd prime with $p \equiv 1$ or $-5 \pmod{12}$ and k be any positive integer, or $p \geq 5$ and k be an even positive integer. Let H_+ be defined as (1). Then, the error correction capacity and the diameter of Cayley graph $\text{Cay}(\mathbb{F}_{q^2}; H_+)$ are 2 and 3, respectively.

Proof: This theorem follows from Theorem 4.7, Theorem 3.1 and Theorem 3.2. \blacksquare

Lemma 5.2: Let $q = p^k$, with p an odd prime. Then, for any $\alpha \in \mathbb{F}_q^\times$, we have

$$\sum_{u \in H_+} \zeta_p^{\text{Tr}_1^{2k}(\alpha u)} = - \sum_{c \in \mathbb{F}_q^\times} \zeta_p^{\text{Tr}_1^k(c + \frac{\mathcal{N}(\alpha)}{c})}.$$

Proof: Let $\alpha = a + \sqrt{\delta}b$, $S = \sum_{u \in H_+} \zeta_p^{\text{Tr}_1^{2k}(\alpha u)}$, and $C_1 = \{(x, y) \in \mathbb{F}_q^2 : x^2 - \delta y^2 = 1\}$. Then

$$S = \sum_{(x, y) \in C_1} \zeta_p^{\text{Tr}_1^{2k}((a + \sqrt{\delta}b)(x + \sqrt{\delta}y))}.$$

From $(a + \sqrt{\delta}b)(x + \sqrt{\delta}y) = ax + \delta by + (ay + bx)\sqrt{\delta}$, we have $\text{Tr}_k^{2k}((a + \sqrt{\delta}b)(x + \sqrt{\delta}y)) = 2ax + 2\delta by$ and

$$\begin{aligned} S &= \sum_{(x, y) \in C_1} \zeta_p^{\text{Tr}_1^k(2ax + 2\delta by)} \\ &= p^{-k} \sum_{(x, y) \in \mathbb{F}_q^2} \zeta_p^{\text{Tr}_1^k(2ax + 2\delta by)} \sum_{c \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^k(c(x^2 - \delta y^2 - 1))} \\ &= p^{-k} \sum_{c \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^k(-c)} \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^k(cx^2 + 2ax)} \sum_{y \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^k(-c\delta y^2 + 2\delta by)} \\ &= p^{-k} \sum_{c \in \mathbb{F}_q^\times} \zeta_p^{\text{Tr}_1^k(-c)} \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^k(cx^2 + 2ax)} \sum_{y \in \mathbb{F}_q} \zeta_p^{\text{Tr}_1^k(-c\delta y^2 + 2\delta by)}. \end{aligned}$$

From Lemma 2.2, we have

$$\begin{aligned} S &= p^{-k} p^{*k} \eta(-\delta) \sum_{c \in \mathbb{F}_q^\times} \zeta_p^{\text{Tr}_1^k(-c + \frac{\delta b^2 - a^2}{c})} \\ &= -\eta_0^k(-1) \eta(-1) \sum_{c \in \mathbb{F}_q^\times} \zeta_p^{\text{Tr}_1^k(c + \frac{a^2 - \delta b^2}{c})} \\ &= - \sum_{c \in \mathbb{F}_q^\times} \zeta_p^{\text{Tr}_1^k(c + \frac{a^2 - \delta b^2}{c})} \quad (\text{from } \eta(-1) = \eta_0^k(-1)) \\ &= - \sum_{c \in \mathbb{F}_q^\times} \zeta_p^{\text{Tr}_1^k(c + \frac{\mathcal{N}(\alpha)}{c})}, \end{aligned}$$

which ends the proof. \blacksquare

Theorem 5.3: Let $q = p^k$, with p an odd prime. Then, $\text{Cay}(\mathbb{F}_{q^2}; H_+)$ is a Ramanujan graph.

Proof: Note that the characters of \mathbb{F}_{q^2} are given by $\chi_\alpha(x) = \zeta_p^{Tr_1^{2k}(\alpha x)}$ as α ranges over \mathbb{F}_{q^2} . From Theorem 2.4, all non-trivial eigenvalues of $\text{Cay}(\mathbb{F}_{q^2}; H_+)$ are $\lambda_{\chi_\alpha} = \sum_{u \in H_+} \chi_\alpha(u)$, with $\alpha \neq 0$. When $\alpha \neq 0$, from Lemma 5.2 and Theorem 2.3, one has

$$|\lambda_{\chi_\alpha}| \leq 2\sqrt{q}.$$

From Lemma 4.1, we have

$$|\lambda_{\chi_\alpha}| \leq 2\sqrt{\#H_+ - 1}.$$

Hence, $\text{Cay}(\mathbb{F}_{q^2}; H_+)$ is a Ramanujan graph. This completes the proof. \blacksquare

Remark When $q = p$, $p \equiv 3 \pmod{4}$, we can identify \mathbb{F}_{p^2} as $\mathbb{F}_p[\sqrt{-1}]$. Thus, $\text{Cay}(\mathbb{F}_{p^2}; H_+)$ are same as the graphs \mathcal{G}_p defined by Camarero and Martínez in [11]. It solves the Conjecture 31 in [11].

B. Cayley graph $\text{Cay}(\mathbb{F}_q^2; H_-)$

Theorem 5.4: Let p be an odd prime with $p \equiv -1$ or $5 \pmod{12}$, k be an odd integer, and $q = p^k > 12$. Let H_- be defined as (2). Then, the error correction capacity and the diameter of subset $\text{Cay}(\mathbb{F}_q^2; H_-)$ are 2 and 3, respectively.

Proof: This theorem follows from Theorem 4.13, Theorem 3.1 and Theorem 3.2. \blacksquare

Theorem 5.5: Let $q = p^k$, with p an odd prime. Then, all non-trivial eigenvalues λ of $\text{Cay}(\mathbb{F}_q^2; H_-)$ satisfy

$$|\lambda| \leq 2\sqrt{\#H_- + 1}.$$

Proof: Note that the characters of \mathbb{F}_q^2 are given by $\chi_{a,b}(x) = \zeta_p^{Tr_1^k(ax+by)}$ as (a,b) ranges over \mathbb{F}_q^2 . From Theorem 2.4, all non-trivial eigenvalues of $\text{Cay}(\mathbb{F}_q^2; H_-)$ are $\lambda_{\chi_{a,b}} = \sum_{(x,y) \in H_-} \chi_{a,b}(x,y)$, with $(a,b) \neq (0,0)$. When $(a,b) \neq (0,0)$, from Theorem 2.3, one has

$$|\lambda_{\chi_{a,b}}| \leq 2\sqrt{q}.$$

From $\#H_- = q - 1$, we have

$$|\lambda_{\chi_\alpha}| \leq 2\sqrt{\#H_- + 1}.$$

This completes the proof. \blacksquare

Remark From Theorem 5.5, Cayley graph $\text{Cay}(\mathbb{F}_q^2; H_-)$ are almost Ramanujan.

VI. 2-QUASI-PERFECT LEE CODES FROM H_+ AND H_-

In this section, we will present the two classes of two-quasi-perfect Lee codes. If $p \equiv -5 \pmod{12}$ and $k = 1$, then -1 is a nonsquare in \mathbb{F}_p^\times and $\mathbb{F}_{p^2} = \mathbb{F}_p[\sqrt{-1}]$. Thus, $\mathcal{N}(x + y\sqrt{-1}) = x^2 + y^2$. In this case, the codes defined in [11] is same as $\mathcal{C}(\mathbb{F}_{p^2}; H_+)$. If $p \equiv 5 \pmod{12}$ and $k = 1$, then -1 is a square in \mathbb{F}_p^\times and $x^2 + y^2 = (x + y\sqrt{-1})(x - y\sqrt{-1})$. In this case, the codes defined in [11] is same as $\mathcal{C}(\mathbb{F}_p^2; H_-)$. Hence, our constructions generalize the constructions presented in [11].

A. Quasi-Perfect Lee Codes $\mathcal{C}(\mathbb{F}_{q^2}; H_+)$

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_{2k}\}$ be a basis of \mathbb{F}_{q^2} over \mathbb{F}_p . Let $H_+ = \{\pm\beta_1, \dots, \pm\beta_n\}$, where $n = \frac{q+1}{2}$ and $\beta_j = \sum_{i=1}^{2k} h_{i,j} \mathbf{e}_i$. Then, from the definition of $\mathcal{C}(\mathbb{F}_{q^2}; H_+)$, the parity-check matrix of $\mathcal{C}(\mathbb{F}_{q^2}; H_+)$ is

$$\mathbf{M}_+ = \begin{pmatrix} h_{1,1} & h_{1,2} & \dots & h_{1,n} \\ h_{2,1} & h_{2,2} & & h_{2,n} \\ \vdots & \vdots & & \vdots \\ h_{2k,1} & h_{2k,2} & & h_{2k,n} \end{pmatrix}. \quad (7)$$

By Theorem 4.6, the rank of \mathbf{M}_+ is $2k$. Thus, the dimension of $\mathcal{C}(\mathbb{F}_{q^2}; H_+)$ is $n - 2k$. From Theorem 4.7, Theorem 3.1 and Theorem 3.2, we have the following theorem.

Theorem 6.1: Let p be an odd prime with $p \equiv 1$ or $-5 \pmod{12}$ and k be any positive integer, or $p \geq 5$ and k be an even positive integer. Let H_+ be defined as (1). Then, the linear code $\mathcal{C}(\mathbb{F}_{q^2}; H_+)$ is a 2-quasi-perfect Lee code over \mathbb{F}_p^n with dimension $\frac{q+1}{2} - 2k$, where $n = \frac{q+1}{2}$. Moreover, \mathbf{M}_+ is the parity-check matrix of $\mathcal{C}(\mathbb{F}_{q^2}; H_+)$.

Example 1: Let $p = 13$, $k = 1$, $n = 7$ and $\mathbb{F}_{13^2} = \mathbb{F}_{13}[\sqrt{2}]$. Then, the codes over \mathbb{Z}_{13}^7 defined by the parity-check matrix

$$\begin{pmatrix} 1 & 9 & 5 & 3 & 10 & 8 & 4 \\ 0 & 1 & 5 & 11 & 11 & 5 & 1 \end{pmatrix}. \quad (8)$$

results in a 2-quasi-perfect 13-ary Lee codes. This codes has $p^{n-2} = 371\,293$ codewords.

Remark If p is an odd prime with $p \equiv -1$ or $+5 \pmod{12}$ and k is an odd integer, $\text{Cay}(\mathbb{F}_{q^2}; H_+)$ only contains $2n^2 + 1$ vertices at distance 2 or less from vertex 0, where $n = \frac{q+1}{2}$. In this case, although $\mathcal{C}(\mathbb{F}_{q^2}; H_+)$ is not a 2-error correcting code, it is very close to it, since only $2n$ syndromes cannot be corrected.

B. Quasi-Perfect Lee Codes $\mathcal{C}(\mathbb{F}_q^2; H_-)$

Let $\{e_1, \dots, e_{2k}\}$ be a basis of \mathbb{F}_q^2 over \mathbb{F}_p . Let $H_- = \{\pm\beta_1, \dots, \pm\beta_n\}$, where $n = \frac{q-1}{2}$ and $\beta_j = \sum_{i=1}^{2k} h_{i,j}e_i$. Then, from the definition of $\mathcal{C}(\mathbb{F}_q^2; H_-)$, the parity-check matrix of $\mathcal{C}(\mathbb{F}_q^2; H_-)$ is

$$\mathbf{M}_- = \begin{pmatrix} h_{1,1} & h_{1,2} & \dots & h_{1,n} \\ h_{2,1} & h_{2,2} & & h_{2,n} \\ \vdots & \vdots & & \vdots \\ h_{2k,1} & h_{2k,2} & & h_{2k,n} \end{pmatrix}. \quad (9)$$

By Theorem 4.12, the rank of \mathbf{M}_- is $2k$. Thus, the dimension of $\mathcal{C}(\mathbb{F}_q^2; H_-)$ is $n - 2k$. From Theorem 4.13, Theorem 3.1 and Theorem 3.2, we have the following theorem.

Theorem 6.2: Let p be an odd prime with $p \equiv -1$ or $5 \pmod{12}$, k be an odd integer, and $q = p^k > 12$. Let H_- be defined as (2). Then, the linear code $\mathcal{C}(\mathbb{F}_q^2; H_-)$ is a 2-quasi-perfect Lee code over \mathbb{F}_p^n with dimension $\frac{q-1}{2} - 2k$, where $n = \frac{q-1}{2}$. Moreover, \mathbf{M}_- is the parity-check matrix of $\mathcal{C}(\mathbb{F}_q^2; H_-)$.

Example 2: Let $p = 23$, $k = 1$ and $n = 11$. Then, the codes over \mathbb{Z}_{23}^{11} defined by the parity-check matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 12 & 8 & 6 & 14 & 4 & 10 & 3 & 18 & 7 & 21 \end{pmatrix}. \quad (10)$$

results in a 2-quasi-perfect 23-ary Lee codes. This codes has $p^{n-2} = 23^9$ codewords.

Remark If p is an odd prime with $p \equiv 1$ or $-5 \pmod{12}$ and $q = p^k > 12$, or, if p is an odd prime with $p \equiv -1$ or $5 \pmod{12}$, k is an even integer and $q = p^k > 12$, $\text{Cay}(\mathbb{F}_q^2; H_-)$ only contains $2n^2 + 1$ vertices at distance 2 or less from vertex 0, where $n = \frac{q-1}{2}$. In this case, although $\mathcal{C}(\mathbb{F}_q^2; H_-)$ is not a 2-error correcting code, it is very close to it, since only $2n$ syndromes cannot be corrected.

Now, Let us give some considerations on the quality of the constructed 2-quasi-perfect Lee codes. Note, that, since the Lee sphere of radius 2 contains $\#B_2^n = 2n^2 + 2n + 1$ words, the graph induced by any 2-quasi-perfect linear code has at least $2n^2 + 2n + 1$ vertices. The graphs $\text{Cay}(\mathbb{F}_q^2; H_+)$ and $\text{Cay}(\mathbb{F}_q^2; H_-)$ constructed in this paper have q^2 vertices. Therefore, for the case $\text{Cay}(\mathbb{F}_q^2; H_+)$, the number of vertices is $q^2 = 4n^2 - 4n + 1 = 2\#B_2^n - 8n + 1$, where $n = \frac{q+1}{2}$. Also, for the case $\text{Cay}(\mathbb{F}_q^2; H_-)$, the number of vertices is $q^2 = 4n^2 + 4n + 1 = 2\#B_2^n - 1$, where $n = \frac{q-1}{2}$. Thus, the reached vertices are asymptotically the double of those that would be reached in the graph associated to a perfect code. In other words, when $q = p$, the density of the codes presented is $\frac{1}{p^2}$.

CONCLUDING REMARKS

In this paper, we firstly survey the relationships between subset sums, Cayley graphs, and Lee linear codes and present some results. Next, we suggest an original approach to construct two classes of 2-quasi-perfect Lee codes defined over the space \mathbb{Z}_p^n for $n = \frac{p^k+1}{2}$ (with $p \equiv 1, -5 \pmod{12}$ and k is any integer, or $p \equiv -1, 5 \pmod{12}$ and k is an even integer) and $n = \frac{p^k-1}{2}$ (with $p \equiv -1, 5 \pmod{12}$, k is an odd integer and $p^k > 12$), where p is an odd prime. To this end, subsets from some quadratic curves over finite fields have been considered. The obtained classes of codes contain the quasi-perfect Lee codes constructed by Camarero and Martínez in [11]. We therefore generalize some results presented in [11] and solve a conjecture proposed in [11] by proving that the related Cayley graphs are Ramanujan or almost Ramanujan. Our approach can be used for constructing other classes of quasi-perfect Lee codes which have many practical applications.

REFERENCES

- [1] J. Astola, Concatenated codes for the Lee metric, IEEE Trans. Inform. Theory, vol. 28, pp. 778-779, 1982.
- [2] J. Astola, On constructing codes and the Lee-metric, Ars Combin., vol. 20-A, pp. 141-152, 1985.
- [3] B. F. AlBdaiwi AlMohammad, On resource placements and fault-tolerant broadcasting in toroidal networks, Ph.D. dissertation, Dept. Comput. Sci., Oregon State Univ., Corvallis, 1997.
- [4] B. AlBdaiwi AlMohammad and B. Bose, I/O placements in 2D toroidal networks, in Proc. IEEE Int. Parallel Processing Symp., pp.431-438, 1998.
- [5] B. F. AlBdaiwi and B. Bose, Quasi-perfect Lee distance codes, IEEE Trans. Inform. Theory, vol. 49, pp. 1535-1539, 2003.
- [6] B. AlBdaiwi AlMohammad, J. H. Kim, and B. Bose, I/O placement strategies and their data locality sensitivities in $k \times k$ toroidal networks, in Proc. HPC Asia'97 Conf. and Exhibition, Seoul, Korea, pp. 343-348, 1997.
- [7] H. Astola and S. Stankovic, On the use of Lee-codes for constructing multiple-valued error-correcting decision diagram, in PROC. 5th Int. Symp. Commun. Control Signal Process.(ISCCSP), May 2012, pp.1-6.
- [8] E. Berlekamp, Algebraic Coding Theory, New York: McGraw-Hill Inc., 1968.
- [9] M. Bae and B. Bose, Resource placement in torus-based networks, IEEE Trans. Comput., vol. 46, no. 10, pp. 1083-1092, 1997.
- [10] B. Bose, R. Broeg, Y. Kwon, and Y. Ashir, Lee distance and topological properties of k -ary n -cubes, IEEE Trans. Comput., vol. 44, pp.1021-1030, 1995.
- [11] C. Camarero and C. Martínez, Quasi-perfect Lee codes of Radius 2 and arbitrarily large dimension, IEEE Trans. Inform. Theory, vol. 62, pp. 1183-1192, 2016.
- [12] J. Chiang and J. Wolf, On channels and codes for the Lee-metric, Inform. Contr., vol. 19, pp. 159-173, 1971.
- [13] S. Golomb and K. Welch, Perfect codes in th Lee metric and the packing of polyominoes, SIAM J. Appl. Math. 18, pp. 302-317, 1970.

- [14] P. Horak and O. Grošek, A new approach toward the Golomb-Welch conjecture. *Eur. J. Combinat.*, vol. 38, pp. 12-22, 2014.
- [15] A. Jiang, M. Schwartz, and J. Bruck, Correcting charge-constrained errors in the rank-modulation scheme, *IEEE Trans. Inf. Theory*, vol. 56, no. 5, pp. 2112-2120, May 2010.
- [16] R. Lidl and H. Niederreiter, Finite fields, in *Encyclopedia of Mathematics*, vol. 30, Cambridge, U.K., Cambridge Univ. Press, 1983.
- [17] M. R. Murty, Ramanujan graphs, *J. Ramanujan Math. Soc.*, vol. 18, no.1, pp. 1-20, 2003.
- [18] C. Q. Queiroz, C. Camarero, C. Martinez, and R. Palazzo, Jr., Quasi-perfect codes from Cauley graphs over integer rings, *IEEE Trans. Inf. Theory*, vol. 59, no. 9, pp. 5905-5916, May 2013.
- [19] R. Roth and P. Siegel, Lee-metric BCH codes and their application to constrained and partial-response channels, *IEEE Trans. Inform. Theory*, vol. 40, pp. 1083-1095, 1994.